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SOJOURNS OF MULTIDIMENSIONAL GAUSSIAN RANDOM FIELDS WITH DEPENDENT COMPONENTS

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Abstract. Asymptotic distributions are considered for functionals

$$|\{x \in v_n(r) : \xi(x)\overline{\in} v_p(f(r))\}|, \quad n \geq 1, p \geq 2,$$

where $v_k(r) = \{x \in \mathbf{R}^k : |x| < r\}$ is a ball in $\mathbf{R}^k$, $|\cdot|$ is the Lebesque measure, $f : (0,\infty) \to (0,\infty)$ is a nonrandom continuous function such that $f(r) \uparrow \infty$, as $r \to \infty$, and $\xi(x) = [\xi_1(x), \cdots, \xi_p(x)]'$ is a homogeneous isotropic vector Gaussian random field with strongly dependent components.

1. INTRODUCTION

In the article we consider limit distribution of sojourns of multidimensional Gaussian random fields with dependent components such that the integral of the correlation function diverges. Problems involving sojourns of vector stationary Gaussian processes with a long range dependence have been studied by Berman [1] and Maejima [6], [7]. The limit theorems for a functional of geometric type of homogeneous isotropic Gaussian random fields with a strong dependence were obtained by Leonenko [3], Leonenko and Ivanov [4] and Leonenko [8].

2. MAIN THEOREMS

Let $\mathbf{R}^n$ be n-dimensional Euclidean space, $\mathcal{B}^n$ the σ-algebra of Borel subsets of $\mathbf{R}^n$, $v_n(r) = \{x \in \mathbf{R}^n : |x| < r\}$ is a ball in $\mathbf{R}^n$, $c_1(n) = 2\pi^{n/2}/[\Gamma(n/2)n]$. We introduce the functions

$$\varphi(u) = (2\pi)^{-\frac{1}{2}}\exp\{-u^2/2\}, \quad u \in \mathbf{R}^1; \quad \Phi(t) = \int_{-\infty}^{t} \varphi(u)du,$$

$$I_\mu(p,q) = \int_0^\mu t^{p-1}(1-t)^{q-1}dt/B(p,q), \quad p > 0, \ q > 0, \mu \in [0,1].$$

Let $\{H_k(u)\}_{k=0}^{\infty}$ be the Hermite polynomials with leading coefficient 1, which form a complete orthonormal system in $L_2(\mathbf{R}^1, \varphi(u)du)$ $(H_0(u) = 1, H_1(u) = u, \ H_2(u) = u^2 - 1, \cdots)$.

Let $\mathcal{L}$ be the class of functions $L(t)$, $t \in (0,\infty)$, slowly varying at infinity and bounded on each finite interval.

(A) Let $\xi(x) = [\xi_1(x), \cdots, \xi_p(x)]'$, $x \in \mathbf{R}^n$ be a measurable mean-square continuous homogeneous and isotropic Gaussian random field with

$$\begin{aligned}
&E\xi(x) = 0, \quad R(|x|) = E\xi(0)\xi(x)' = (R_{ij}(|x|))_{1\leq i,\, j\leq p},\\
&R_{ii}(|x|) = a(|x|), \quad i = 1, \cdots, p; \quad R_{ij}(|x|) = b(|x|), \quad i \neq j; i,j = 1, \cdots, p;\\
&a(0) = 1, \quad b(0) = \rho_0 \in [0,1),\\
&a(|x|) \sim L(|x|)/|x|^\alpha, \quad b(|x|) \sim \rho_\infty L(|x|)/|x|^\alpha, \quad |x| \to \infty,\\
&\rho_\infty \in [0,1), \quad L \in \mathcal{L}, \quad \alpha > 0.
\end{aligned}$$

We introduce the indicator function $1\{\cdot\}$. Consider the functional

$$G(r) = |\{x \in v_n(r) : \xi(x)\overline{\in} v_p(f(r))\}| = \int_{v_n(r)} 1\{\xi(x) \in \mathbf{R}^p \backslash v_p(f(r))\}dx,$$

where $f(r)$, $r > 0$ is a real nonrandom continuous function such that $\lim f(r) = \infty, r \to \infty$.

The limit distributions of the random variables $G(r)$ as $r \to \infty$ can be given with the help of multiple stochastic integrals (m.s.i.) (see, for example, Major [8]).

Let $F(\cdot)$ be a non-atomic spectral measure of a homogeneous (may be generalized) random field. Let $Z_F(\cdot)$ be a complex Gaussian spectral random measure subordinate to $F(\cdot)$ $(EZ_F(A) = 0,$ $Z_F(A) = \overline{Z_F(-A)},$ $EZ_F(A)\overline{Z_F(B)} = F(A \cap B),$ $A, B, \in \mathcal{B}^n)$.

Let $L_2(\mathbf{R}^{nm}, F^m)$ be a Hilbert space of symmetric complex-valued functions $f(\lambda_1, \cdots, \lambda_m)$, $\lambda_j \in \mathbf{R}^n$, $j = 1, \cdots, m$, such that

$$f(\lambda_1, \cdots, \lambda_m) = \overline{f(-\lambda_1, \cdots, -\lambda_m)}; \qquad \int\limits_{\mathbf{R}^{nm}} | f(\lambda_1, \cdots, \lambda_m) |^2 \prod_{j=1}^{m} F(d\lambda_j) < \infty.$$

Then the m.s.i.

$$S_m(f) = \int\limits_{\mathbf{R}^{nm}}{}' f(\lambda_1, \cdots, \lambda_m) \prod_{j=1}^{m} Z_F(d\lambda_j)$$

is defined as an isometric mapping of the space $L_2(\mathbf{R}^{nm}, F^m)$ into $L_2(P)$ $(f \to S_m(f))$, constructed first on the set of "simple" functions in such a way that integration over the "hyperplanes" $\lambda_i = \pm\lambda_j$, $i, j = 1, \cdots, m$, is excluded (see Major [8]). Let

$$I_\nu(z) = \sum_{k=0}^{\infty} (-1)^k z^{2k+\nu} / [2^{2k+\nu} k! \Gamma(k + \nu + 1)], \quad \nu > -\frac{1}{2}$$

be the ν order Bessel function of the first kind. We need the following assertion (Leonenko, Ivanov [4]).

THEOREM 1. *Let $\zeta(x)$, $x \in \mathbf{R}^n$, be real measurable homogeneous isotropic mean-square continuous Gaussian field with $E\zeta(x) = 0, E\zeta^2(x) = d_1$, and*

$$E\zeta(0)\zeta(x) = \int\limits_{\mathbf{R}^n} e^{i<\lambda,x>} F(d\lambda) \sim d_1 L(| x |) / | x |^\alpha, \quad 0 < \alpha < n, \quad \text{as } | x | \to \infty,$$

where $d_1 > 0, L \in \mathcal{L}$.

Then the measures $F_r(A) = d_1^{-1} L^{-1}(r) r^{-\alpha} F(r^{-1}A), A \in \mathcal{B}^n$ converge locally weakly as $r \to \infty$ to a locally finite measure $F_0(A), A \in \mathcal{B}^n$, that satisfies the condition: $F_0(A) = s^{-\alpha} F_0(sA)$, $s \in (0, \infty)$, and is determined by the Fourier transform

$$\int\limits_{\mathbf{R}^n} e^{i<z,\lambda>} I^2_{\frac{n}{2}}(| \lambda |) | \lambda |^{-n} F_0(d\lambda) = \int\limits_{v_n(1)} \int\limits_{v_n(1)} | x - y + z |^{-\alpha} dx dy.$$

If $\alpha \in (0, n/m)$, $n \geq 1, m \geq 1$, then as $r \to \infty$ the finite-dimensional distributions of the random processes

$$Y_r(t) = d_1^{-m/2} r^{m\alpha/2 - n} L^{-m/2}(r) \int\limits_{v_n(rt^{1/n})} H_m(\zeta(x)) dx, \quad t \in [0, 1]$$

weakly converge to the finite-dimensional distributions of the process

$$W^{(m)}(t) = (2\pi)^{n/2} t^{1/2} \int\limits_{\mathbf{R}^{nm}}{}' \frac{I_{\frac{n}{2}}(| \lambda_1 + \cdots + \lambda_m | t^{1/n}}{| \lambda_1 + \cdots + \lambda_m |^{n/2}} \prod_{j=1}^{m} Z_{F_0}(d\lambda_j), \quad t \in [0, 1], \tag{2.1}$$

where $Z_{F_0}(\cdot)$ is a random measure subordinate to a measure $F_0(\cdot)$.

Theorem 1 is a variant of a non-central limit theorem of Dobrushin and Major [2].

Let

$$d_1 = \frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}, \quad d_2 = \cdots = d_p = \frac{(1-\rho_\infty)}{(1-\rho_0)}, \tag{2.2}$$

$$\mu_1 = [1+(p-1)\rho_0]^{-1/2}, \quad \mu_2 = \cdots = \mu_p = (1-\rho_0)^{-1/2} \tag{2.3}$$

and

$$K_\star = (\mu_1^2 - \mu_2^2)/2 \geq 0.$$

If $K_\star = 0, \quad \alpha \in (0, n/2)$, then

$$\text{Var } G(r) \sim A_1^2(r) = c_2(n,2,\alpha)c_1^2(p)(2\pi)^{-p}(2p)e^{-f(r)}[f(r)]^p r^{2n-2\alpha}L^2(r) \tag{2.4}$$

as $r \to \infty$, where

$$c_2(n,m,\alpha) = m!2^{n-m\alpha+1}\pi^{n-\frac{1}{2}}\Gamma(\frac{n-m\alpha+1}{2})\times \tag{2.5}$$

$$\times \left[(n-m\alpha)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n-m\alpha+2}{2}\right)\right]^{-1}, \quad n \geq 1, \quad m \geq 1, \quad 0 < \alpha < n/m.$$

If $K_\star > 0, \alpha \in (0, n/2), p \geq 2$, then

$$\text{Var } G(r) \sim A_2^2(r) = c_2(n,2,\alpha)(p-1)^2\mu_1^2\mu_2^{2(p-1)}K_\star^{1-p}d_2^2\pi 2^{p-1}f(r)r^{2n-2\alpha}L^2(r) \tag{2.6}$$

as $r \to \infty$.

THEOREM 2. *Suppose that assumption (A) for $\alpha \in (0, \frac{n}{2})$ hold and $f(r) = o(\ln\ r)$ as $r \to \infty$.*
If $K_\star = 0$, then the limit distribution of the random variables $[G(r) - EG(r)]/A_1(r)$ coverge in distribution as $r \to \infty$ to the distribution of the random variable

$$\sum_{j=1}^{p} W_j^{(2)}(1)/\sqrt{2pc_2(n,2,\alpha)}$$

where $W_j^{(2)}(1), \quad j = 1, \cdots, p$ are independent copies of $W^{(2)}(t), t = 1$, given by (1.1) and $c_2(n,2,\alpha)$ given by (2.5).
If $K_\star > 0, p \geq 2$, then the distribution of the random variables $[G(r) - EG(r)]/A_2(r)$ converge in distribution to the distribution of the random variable

$$\sum_{j=2}^{p} W_j^{(2)}(1)/\sqrt{2c_2(n,2,\alpha)(p-1)},$$

where $A_2^2(r)$ is given by (2.6) and $W_j^{(2)}(1), \quad j = 2, \cdots, p$, are independent copies of $W^{(2)}(1)$.

3. PROOF OF THEOREM 2

Let $V =$ diag $(d_1, \cdots, d_p)$ be a diagonal matrix, where $d_j, j = 1, \cdots, p$, are given by (2.2) and $T = (t_{ij})_{1 \leq i,j \leq p}$ is a matrix such that

$$\begin{aligned}
t_{1j} &= [p(1+(p-1)\rho_0]^{1/2}, 1 \leq j \leq p;\\
t_{ij} &= [i(i-1)(1-\rho_0)]^{-1/2}, 2 \leq i \leq p, \quad 1 \leq j \leq i;\\
t_{ii} &= -(i-1)[i(i-1)(1-\rho_0)]^{-1/2}, \quad 2 \leq i < p;\\
t_{ij} &= 0, \quad 2 \leq i \leq p-1, i < j \leq p.
\end{aligned}$$

Note that $T^{-1} = (s'_{ij})_{1\le i,j\le p}$, $s_{i1} = \sqrt{1+(p-1)\rho_0)/p}$, $1 \le i < p$;

$s_{ij} = \sqrt{(1-\rho_0)/i(i-1)}$, $2 \le i < p$, $1 \le i \le j$; $s_{ii} = -(i-1)\sqrt{(1-\rho_0)/i(i-1)}, 2 \le i \le p$;

$s_{ij} = 0$, $2 \le i \le p-1, j < i \le p$.

Lemma 1. If the condition (A) holds, then the random field $\eta(x) = [\eta_1(x), \cdots, \eta_p(x)]' = T\xi(x), x \in \mathbf{R}^n$, has independent components, i.e. $E\eta(x) = 0$,

$$\tilde{R}(\mid x \mid) = (\tilde{R}_{ij}(\mid x \mid))_{1\le i,j\le p} = E\eta(0)\eta(x)' = V(L(\mid x \mid)/\mid x \mid^{\alpha}) \text{ as } \mid x \mid \to \infty,$$

$L \in \mathcal{L}$, $\alpha > 0$; $\tilde{R}_{ii}(0) = 1$.

The proof of Lemma 1 is based on the procedure of orthogonalization.

Lemma 2. Let $(\xi_1, \cdots, \xi_{2p})$ be $2p$ -dimensional Gaussian vector with $E\xi_j = 0$, $E\xi_j^2 = 1$, $1 \le j \le 2p$, $E\xi_j\xi_{j+p} = r_j$, $1 \le j \le p$; $E\xi_j\xi_k = 0$, $(j,k)\bar{\in}\{(i,j) : (j+p, j+p), (j, j+p), 1 \le i \le p\}$. Then

$$E\prod_{j=1}^{p} H_{k_j}(\xi_j)H_{m_j}(\xi_{j+p}) = \prod_{j=1}^{p} \delta_{k_j}^{m_j} k_j! r_j^{k_j}$$

where δ_m^k is the Kronecker symbol.

PROOF. See Taqqu [9].

Lemma 3. Let $\bar{\alpha}$ and $\bar{\beta}$ be random points chosen in $v_n(r)$ independetly according to the uniform law. Then the density of the distribution of the distance $\rho = \mid \bar{\alpha} - \bar{\beta} \mid$ between $\bar{\alpha}$ and $\bar{\beta}$ has the form

$$p_\rho(z) = nr^{-n}z^{n-1}I_{\mu(z,r)}\left(\frac{n+1}{2}, \frac{1}{2}\right), \quad 0 \le z \le 2r, \quad \mu(z,r) = 1 - \left(\frac{z}{2r}\right)^2.$$

PROOF. See Santalo [10].

If $f(\rho), \rho \ge 0$, is a Borel function, then by Lemma 3

$$\int_{v_n(r)}\int_{v_n(r)} f(\mid x - y \mid)dxdy = \mid v_n(r) \mid^2 Ef(\mid \bar{\alpha} - \bar{\beta} \mid) =$$

$$= \frac{4\pi^n}{n\Gamma^2(\frac{n}{2})} r^n \int_0^{2r} z^{n-1} f(z) I_{\mu(z,r)}\left(\frac{n+1}{2}, \frac{1}{2}\right) dz. \tag{3.1}$$

Let $\nu = (k_1, \cdots, k_p)$ be a multi-index, $u = (u_1, \cdots, u_p) \in \mathbf{R}^p, E_\nu(u) = \prod_{j=1}^p H_{k_j}(u_j)$. Then the polynomials $\{E_\nu(u)\}_\nu$ form complete orthonormal system in the Hilbert space

$$L_2(\mathbf{R}^p, \Phi^p) = \left\{ g(u), u \in \mathbf{R}^p : \int_{\mathbf{R}^p} g^2(u)\left\{\prod_{j=1}^p \phi(u_j)\right\}du < \infty \right\}.$$

Let $k \ge 0$ be an integer, $S_k = \{\nu = (k_1, \cdots, k_p) : k_j \ge 0, \quad 1 \le j \le p, \sum_{j=1}^p k_j = k\}$. The function $g_r(u) \in L_2(\mathbf{R}^p, \Phi^p)$ can be expanded in Fourier series:

$$g_r(u) = \sum_{k\ge 0}\sum_{S_k} C_r(\nu)E_\nu(u) = \sum_{k\ge 0}\sum_{(k_1,\cdots,k_p)\in S_k} C_r(k_1, \cdots, k_p)E_\nu(u), \tag{3.2}$$

where

$$C_r(\nu) = C_r(k_1, \cdots, k_p) = \left(\prod_{1\le j\le p} k_j!\right)^{-1} \int_{\mathbf{R}^p} g_r(u)E_\nu(u)\left\{\prod_{j=1}^p \phi(u_j)\right\}du.$$

(B) Let $g_r(u) \in L_2(\mathbf{R}^p, \Phi^p)$ and there exist an integer $m \geq 1$ such that $C_r(k_1, \cdots, k_p) = C_r(\nu) = 0$ if $\nu = (k_1, \cdots, k_p) \in S_k$, $1 \leq k \leq m-1$, but $C_r(m_1, \cdots, m_p) \neq 0$ for some $(m_1, \cdots, m_p) \in S_m$, $m_1 + \cdots + m_p = m$.

Let $\eta(x) \in \mathbf{R}^p, x \in \mathbf{R}^n$ be a random field constructed by Lemma 1 and $g_r(u) \in L_2(\mathbf{R}^p, \Phi^p)$. We consider the random variables

$$K_r(t) = \int\limits_{v_n(rt^{1/n})} g_r(\eta(x))dx, \quad t \in [0,1];$$

$$L_r^{(m)}(t) = \sum_{S_m} C_r(\nu) \int\limits_{v_n(rt^{1/n})} E_\nu(\eta(x))dx, \quad t \in [0,1],$$

where $g_r(\eta(x)) = g_r(\eta_1(x), \cdots, \eta_p(x))$.

Lemma 4. Let assumptions $(A), (B)$ hold for $\alpha \in (0, \frac{n}{m})$ and

$$\zeta_r = \left\{ \sum_{S_m} C_r^2(m_1, \cdots, m_p) \right\}^{-1} \int\limits_{\mathbf{R}^p} g_r^2(u) \left\{ \prod_{j=1}^{p} \phi(u_j) \right\} du = o\left(\frac{r^\alpha}{L(r)}\right) \tag{3.3}$$

as $r \to \infty$. Then the finite-dimensional distributions of the random process

$$X_r(t) = [K_r(t) - EK_r(t)]/\sqrt{\text{Var } K_r(1)}, \quad t \in [0,1]$$

are the same as the limit distributions of the random process

$$X_r^{(m)}(t) = L_r^{(m)}(t)/\sqrt{\text{Var } L_r^{(m)}(1)}, \quad t \in [0,1]$$

(if one of these exists).

The proof of Lemma 4 is similar to Maejima [6], using (3.1), (3.2) and Lemma 2.

Lemma 5. Let $\alpha > 0$, $\beta > 0$, $0 < a \leq \infty$ and $f \in C([0,d], \mathbf{R})$. If $f(x) = f(0) + O(x)$ as $x \to 0$ then

$$\int_0^a x^{\beta-1} f(x) e^{-\lambda x^\alpha} dx = \frac{1}{\alpha} f(0) \Gamma\left(\frac{\beta}{\alpha}\right) \lambda^{-\frac{\beta}{\alpha}} + O\left(\lambda^{-\frac{\beta+1}{\alpha}}\right)$$

as $\lambda \to \infty$.

PROOF. See Zorich [11].

Let $\tilde{\Delta}_r = \{y \in \mathbf{R}^p : T^{-1}y \in \Delta_r\}$. Then

$$G(r) = \int\limits_{v_n(r)} 1\{\xi(x) \in \Delta_r\} dx = \int\limits_{v_n(r)} 1\{\eta(x) \in \tilde{\Delta}_r\} dx,$$

where $\eta(x), x \in \mathbf{R}^n$ is a vector random field with independent components (see Lemma 1.)

Let $\Delta_r = \mathbf{R}^p \backslash v_p(f(r))$. Then

$$\tilde{\Delta}_r(f(r)) = \left\{ (y_1, \cdots, y_p)' : \sum_{j=1}^{p} \left(\frac{y_j}{\mu_j}\right) < f(r) \right\},$$

where $\mu_j, j = 1, \cdots, p,$ are given by (2.3).

We consider the case $K_\star = 0$, i.e. $\rho_0 = 0$ (see condition (A)). If $g_r(u) = 1\{|u|^2 > f(r)\}$, then $C_r(\nu) = 0, \nu = (k_1, \cdots, k_p) \in S_1$, i.e. $k_1 + \cdots + k_p = 1$ or $(k_1, \cdots, k_p) \in S_2$, but $k_i = k_j = 1$ for some $i \neq j$. If $K_\star = 0$, then

$$C_r(2, 0, \cdots, 0) = c_1(p)(2\pi)^{-p/2} e^{-f(r)/2} [f(r)]^{p/2},$$
$$C_r(0, 2, 0, \cdots, 0) = \cdots = C_r(0, \cdots, 0, 2) = C_r(2, 0, \cdots, 0).$$

The condition (B) holds for $m = 2$,

$$\int_{\mathbf{R}^p} g_r^2(u)\left\{\prod_{j=1}^{p}\phi(u_j)\right\}du \sim c_4[f(r)]^{(p-1)/2}e^{-f(r)/2}, \quad c_4 > 0$$

as $r \to \infty$.

Using (3.2) we have ($c_5 > 0, c_6 > 0$)

$$\zeta_r \sim c_5[f(r)]^{p-1}e^{-f(r)/2}/[f(r)]^p e^{-f(r)} = c_6 e^{f(r)/2}/f(r) = o\left(\frac{r^\alpha}{L(r)}\right).$$

Then by Lemma 4 the limit distribution of the random variables $[G(r) - EG(r)]/A_1(r)$ as $r \to \infty$ is the same that as that of the random variables

$$C_r(2,0,\cdots,0)\sum_{j=1}^{p}\int_{v_n(r)} H_2(\eta_j(x))dx/[c_2(n,2,\alpha)2pC_r(2,0,\cdots,0)r^{2n-2\alpha}L^2(r)]^{1/2} =$$

$$= \sum_{j=1}^{p}\int_{v_n(r)} H_2(\eta_j(x))dx/\sqrt{c_2(n,2,\alpha)r^{2n-2\alpha}L^2(r)2p}. \tag{3.4}$$

Using Theorem 1 with $m = 2, t = 1$ we have that the limit distribution has the form $\sum_{j=1}^{p} W_j^{(2)}(1)/\sqrt{2c_2(n,2,\alpha)p}$.

If $K_\star > 0$, then $C_r(k_1,\cdots,k_p) = 0$ if $(k_1,\cdots k_p) \in S_1$ or $(k_1,\cdots,k_p) \in S_2$ but $k_i = k_j = 1$ for some $i \neq j$.

We have

$$C_r(2,0,\cdots,0) = \frac{1}{2}\int_{\tilde{\Delta}_3(r)} H_2(u_1)\left\{\prod_{j=1}^{p}\phi(u_j)\right\}du =$$

$$= -\mu_1\mu_2^{p-1}\int_{y_2^2+\cdots+y_p^2\le f(r)}\prod_{j=2}^{p}\phi(\mu_2 y_j)dy_j\int_0^{\sqrt{f(z)-\sum_{j=2}^{p}y_j}} H_2(\mu_1 y_1)\phi(\mu_2 y_1)dy_1 =$$

$$= (2\pi)^{-p/2}\mu_1\mu_2^{p-1}\int_{y_2^2+\cdots+y_p^2\le f(r)} e^{-\mu_2^2(y_2^2+\cdots+y_p^2)/2}\times$$

$$\times e^{-\mu_1^2(f(r)-(y_2^2+\cdots+y_p^2)/2}\sqrt{f(r)-\sum_{j=2}^{p}y_j^2}\quad dy_2\cdots dy_p =$$

$$= (2\pi)^{-p/2}\mu_1\mu_2^{p-1}e^{-\mu_1^2 f(r)/2}2\pi^{(p-1)/2}\Gamma^{-1}\left(\frac{p-1}{2}\right)\int_0^{\sqrt{f(r)}}\rho^{p-2}e^{-K_\star\rho^2}\sqrt{f(r)-\rho^2}\quad dp =$$

$$= (2\pi)^{-p/2}\mu_1\mu_2^{p-1}e^{-\mu_1^2 f(r)/2}c_1(p)[f(r)]^{p/2}\int_0^1 x^{p-2}e^{-x^2K_\star f(r)}\sqrt{1-x^2}\quad dx.$$

By Lemma 5

$$\int_0^1 x^{p-2}e^{-x^2\lambda}\sqrt{1-x^2}\quad dx \sim \Gamma\left(\frac{p-1}{2}\right)\lambda^{-(p-1)/2}/2, \quad \lambda \to \infty.$$

Then
$$C_r(2,0,\cdots,0) \sim \mu_1\mu_2^{p-1}(2\pi)^{-p/2}\pi^{(p-1)/2}K_*^{-(p-1)/2}\sqrt{f(r)}e^{-\mu_1^2 f(r)/2}$$
as $r \to \infty$.
Analogously
$$C_r(0,2,0,\cdots,0) = C_r(0,0,2,0,\cdots,0) = \cdots = C_r(0,0,\cdots,0,2) \sim$$
$$\sim \mu_1\mu_2^{p-1}(2\pi)^{-p/2}\pi^{(p-1)/2}K_*^{-(p-1)/2}\sqrt{f(r)}e^{-\mu_2^2 f(r)/2}, \quad r \to \infty.$$
Therefore
$$\mathrm{Var}G(r) \sim c_2(n,2,\alpha)r^{2n-2\alpha}L^2(r)2(C_r^2(2,0,\cdots,0)d_1^2 + (p-1)d_2^2C_r^2(0,2,0,\cdots,0)] \sim$$
$$\sim c_2(n,2,\alpha)r^{2n-2\alpha}L^2(r)(p-1)d_2^2C_r^2(0,2,0,\cdots,0), \quad r \to \infty,$$
bacause of
$$C_r(2,0,\cdots,0)/C_r(0,2,0,\cdots,0) = e^{-K_*f(r)} \to 0 \tag{3.5}$$
as $r \to \infty$.

The conditions (3.2) hold bacause of
$$\zeta_r \sim c_7[(f(r))^{p-1}e^{-f(r)/2}]/[f(r)e^{-\mu_2^2 f(r)}] =$$
$$= c_7\exp\{-[(\mu_2^2 - 1/2)f(r) - (p-2)\log f(r)]\} = o(r^\alpha/L(r))$$
as $r \to \infty (c_7 > 0)$.

By Lemma 4 with $m = 2, t = 1$, the limit distribution of the random variable $[G(r) - EG(r)]/A_2(r)$ is the same as that of the random variable
$$\left[C_r(2,0,\cdots,0)\int\limits_{v_n(r)} H_2(\eta_1(x))dx + C_r(0,2,0,\cdots,0)\sum_{j=2}^{p}\int\limits_{v_n(r)} H_2(\eta_j(x))dx\right] \times$$
$$\times\, [c_2(n,2,\alpha)r^{2n-2\alpha}L^2(r)(p-1)d_2^2C_r(0,2,0,\cdots,0)]^{1/2}. \tag{3.6}$$
Using (3.5) we have
$$\mathrm{Var}\left[C_r(2,0,\cdots,0)\int\limits_{v_n(r)} H_2(\eta_1(x))dx/r^{n-\alpha}L(r)d_2\times\right.$$
$$\left.\times C_r(0,2,0,\cdots,0)\sqrt{c_2(n,2,\alpha)(p-1)}\right] \to 0$$
as $r \to \infty$. Then the limit distribution of $[G(r) - EG(r)]/A_2(r)$ is the same as that of the second term of (3.6) i.e.
$$\sum_{j=2}^{p}\int\limits_{v_n(r)} H_2(\eta_j(x))dx/\sqrt{c_2(n,2,\alpha)(p-1)}d_2r^{n-\alpha}L(r). \tag{3.7}$$
Using Theorem 1 we have that there exists the limit distribution of (3.7) for $\alpha \in (0, \frac{n}{2})$ and this limit distribution has the form
$$\sum_{j=2}^{p} W_j^{(2)}(1)/\sqrt{2c_2(n,2,\alpha)(p-1)}, \quad p \geq 2.$$

Theorem 2 is proved.

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